UNDERSTANDING PROBABILITY

OVERVIEW

There are many situations when one needs to know the likelihood of something happening. Consider Weather forecasting. One typically likes to know whether it is likely to rain today or over the next few days. This helps in planning and scheduling outdoor events and activities. A life Insurance agent needs to know likely or average life expectancy of various groups based on their occupation in order to arrive at an estimate of premium rates. Quality control in manufacturing requires analysis of expected or likely defect rates in a production process.

This means, often, one needs a measure for dealing with uncertainty – in other words, we use probability to describe the likelihood of something happening. In the simplest sense,

$$Probability\ of\ an\ event\ happening = \frac{Number\ of\ favourable\ outcomes}{Total\ number\ of\ outcomes}$$

Examples:

EXAMPLE-1

Consider tossing a coin. There are two outcomes in this case – Head (H) or Tail (T). Therefore, probability of getting a Head or Tail is 0.5 (1/2)

Probability predicts the likelihood of outcomes. If we were to toss the coin 100 times, we can expect 50 heads to turn up. The actual number will be close to 50. It can be 48, 53 etc.

Experiments, Trials and Sample Space

In this example, the experiment, which is repeatable, is tossing of the coin. A trial is a single performance of the experiment – 100 tosses of the coin will be 100 trials.

Sample Space is defined as set of all possible outcomes. Therefore, the sample space is {H,T}

Equally likely outcomes

It is assumed we are tossing a fair coin, and the outcomes, i.e., H, T are equally likely. This will not be the case if the coin is biased – in this case, more heads or more tails will turn up over a series of experiments.

Probability Range

By definition, range of values for probability lies between 0 and 1.
If there are no favourable outcomes, the event cannot happen and the probability is 0.
If all outcomes are favourable, the probability of the event is 1, i.e., we can say the event will occur every time a trial is performed, with 100% certainty.

EXAMPLE -2

Consider tossing four coins at the same time. What is the probability of getting three or more heads?

Solution

There are 16 outcomes for this example.

Sample Space = {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT }

The number of favorable outcomes corresponding to the event, i.e., getting three or more heads is 5.

Therefore, required probability = 5/16

Mutually Exclusive Events

In example 2, we calculated the probability for getting three or more heads. Let's call this Event 1. Suppose we define Event 2 as getting two heads or less. Obviously, these two Events cannot happen at the same time and therefore they are mutually exclusive events.

If we re-define Event 2 as getting three heads or less, the two Events will not be mutually exclusive. This is because for outcomes having exactly three heads, the Event is classified as Event-1 as well as Event-2.

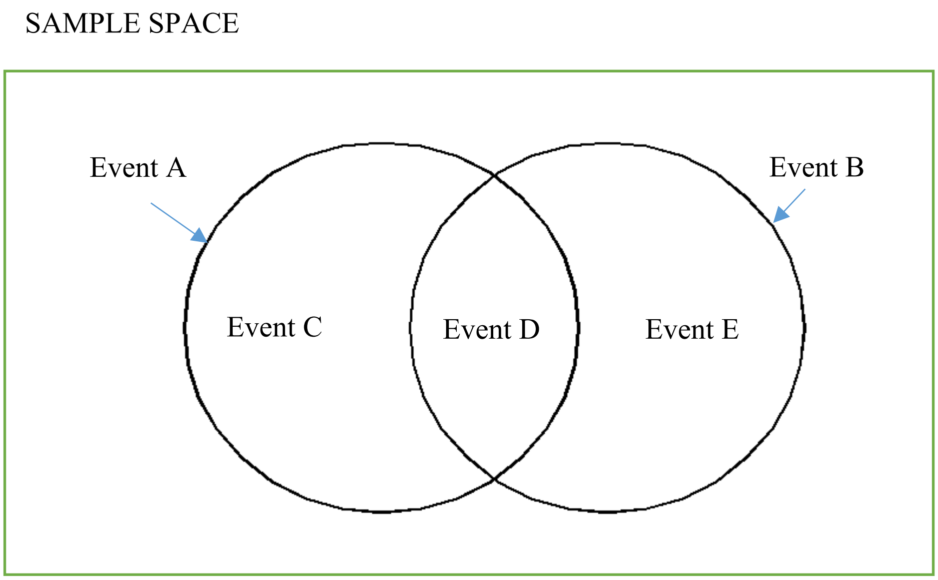
Complementation of Events

In example 2, if Event 1 was getting three or more heads and Event 2 was getting two heads or less, they are mutually exclusive as well as complementary. It means, together, they account for all outcomes in the sample space in a series of experiments. Therefore, it follows:

$$Even\ 2 = Event\ 1's\ complementP(Event\ 2) = Event\ 1's\ complement = 1 - P(Event\ 1)$$

Venn Diagrams

Venn diagrams facilitate the analysis of probabilities. They pictorially depict various Events and how they overlap within the sample space



Union and Intersection of Events

Let us denote Event A = A, Event B = B, Event C = C, Event D = D, Event E = E. The probability of intersections of A and B is defined as follows

$$P(A \cap B) = P(D)$$

The probability of Union of A and B is defined as

$$P(A \cup B) = P(C) + P(D) + P(E)$$

The probabilities of A and B respectively, are

$$P(A) = P(C) + P(D); P(B) = P(E) + P(D)$$

Therefore, it follows:

$$P(A \cup B) = P(A) + P(B) - P(D) = P(A) + P(B) - P(A \cap B)$$

If $P(A \cap B) = \emptyset$, the events A and B are mutually exclusive, i.e, $P(A \cup B) = P(A) + P(B)$

EXAMPLE-3

In tossing a fair die, what is the probability of getting an even number or a number less than 4?

Solution

Let Event A be getting an even number. The sample space = {1, 2, 3, 4, 5, 6}, favourable outcomes for Event A = {2, 4, 6}. If Event B is getting a number less than 4, favourable outcomes for Event B = {1, 2, 3}. The favourable outcomes for the event $A \cap B$ = {2}. It then follows

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ = 3/6 + 3/6 – 1/6 = 5/6

Permutations and Combinations

Consider the following Quality Control problem. A lot of 100 items has 10 defective items. If 5 items are picked from the lot, what is the probability of none of them being defective?

To solve this problem, consider the sample space. Sample Space = Number of ways of selecting 5 items from 100 items. This will be 100 x 99 x 98 x 97 x 96 = 9.034,502,400 ways. A very large number indeed!

The desired event is none of the five items picked are defective. The number of selections = 90 x 89 x 88 x 87 x 86 = 5,273,912,160 ways, an equally large number!

Required Probability = 5,273,912,160/ 9.034,502,400 = 0.58

Permutations and Combinations provide a more systematic way of counting number of points in a sample space and evaluating probabilities for such cases with considerable simplification of the arithmetic involved.

Permutations

Different elements

Consider the word “cat”. It has three different letters. ‘c’. ‘a’, ‘t’. There are six different ways (3 x2x1) to arrange the letters in various orders, i.e, *cat, cta, act, atc, tac, tca.* This is referred to as permutations of three different items.

Permutations of n different elements can be done in n factorial ways where n! = n x (n-1) x (n-2) x ….3 x 2 x 1

Groups of identical elements

Consider three red and two blue balls. There are 10 different ways to arrange the balls, i.e. BRBRR, BRRBR, BRRRB, RBRBR, RBRRB, RRBRB, BBRRR, RBBRR, RRBBR, RRRBB. This in fact, is, 5!/(3! X 2!)

In general, number of permutations of *n* elements with *k* groups of $n_1, n_2 \ldots\ldots n_k$ identical elements, where $n = n_1, + n_2 + \ldots\ldots + n_k$ is given by $n! /(n_1! \times n_2! \times \ldots \times n_k!)$

Permutations of n elements taken k at a time (no repetitions)

Consider four digits 1, 2, 3, 4. There are 12 permutations of 2 digits out of the four. These are 12, 21, 13, 31, 14, 41, 23, 32, 24, 42, 34, 43 i.e (4! / 2!)

In general, number of permutations of n elements taken k at a time with no repetitions is n! / (n-k)!

Permutations of n elements taken k at a time (with repetitions)

There are 16 permutations of 2 digits out of four digits 1, 2, 3, 4 with repetitions. These are 11, 12, 21, 13, 31, 13, 41, 22, 23, 32, 24, 42, 33, 34, 43, 44 i.e., 4 x 4 or 4^2.

In general, number of permutations of n elements taken k at a time with repetitions is n^k.

Combinations

Unlike permutations, combinations refers to selection of elements without reference to order.

Combinations of n elements taken k at a time (no repetitions)

Consider the example of four digits 1, 2, 3, 4. There are 6 combinations of 2 digits out of the four. These are 12, 13, 14, 23, 24, 34, i.e 4! / (2! X 2!)

In general, number of combinations of n elements taken k at a time with no repetitions is n! / {(n-k)! k!}

Combinations of n elements taken k at a time (with repetitions)

There are 10 combinations of 2 digits out of four digits 1, 2, 3, 4 with repetitions. These are 11, 12, 13, 14, 22, 23, 24, 33, 34, 44

In general, number of combinations of n elements taken k at a time with repetitions is (n+k-1)! / {(n-1)! k!}

Joint Probability of Independent Events

If events A and B are such that $P\ (A \cap B) = P(A)P(B)$, they are called independent events. This means the probability of A does not depend on the occurrence or non-occurrence of B and vice-versa.

EXAMPLE-4

A box contains hundred screws out of which five are defective. What is the probability of drawing two defective screws with replacement?

Solution

Probability of drawing a defective screw is 5/100 = 0.05.

P(A) = Probability of picking defective screw during first draw (Event A) = 0.05.
P(B) = Probability of picking defective screw during second draw (Event B) = 0.05.

Since the events are independent, $P\ (A \cap B) = 0.05\ x\ 0.05 =\ 0.0025$

Conditional Probability

Often it is required to find the probability of Event B after Event A has occurred in a reduced Sample Space. This is called conditional probability and defined as $P(B|A) = P(A \cap B)/P(A)$. Note events A and B are no longer independent.

EXAMPLE-5

In box contains hundred screws out of which five are defective. What is the probability of drawing two defective screws without replacement?

Solution

Probability of drawing a defective screw is 5/100 = 0.05.

P(A) = Probability of picking defective screw during first draw (Event A) = 0.05.
P(B) = Probability of picking defective screw during second draw (Event B) = 4/99 = 0.04

Therefore, $P(A \cap B) = P(B|A)P(A) = 0.05\ x\ 0.04 =\ 0.002$

Conditional Probability and Multiplication Rule (also called Bayes Theorem)

This rule simply states that the conditional probability definitions of events A and B are interchangeable:

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

EXAMPLE-6

In a box containing hundred screws, 10 are too short, 5 are too slim and 2 of these 15, are both slim and short. What are the conditional probabilities of picking a short screw that is also slim as well as picking a slim screw that is also short.

Solution

2 screws in the lot are both slim and short. Therefore, $P(A \cap B)$ = 2/100 = 0.02
Let event A represent drawing a short screw from the lot. P(A) = 10/100 = 0.1
Let event B represent drawing a slim screw from the lot. P(B) = 5/100 = 0.05

P(A|B) = Conditional Probability that a short screw that is also slim = $P(A \cap B)$ / P(B) = 0.02/0.05 = 0.4
P(B|A) = Conditional Probability that a slim screw that is also short = $P(A \cap B)$ /P(A) = 0.02/0.1 = 0.2

Outcomes to random Variables (RV)

Experimental outcomes depending on chance are quantified by converting them to random/ stochastic variables. For example, in a production process, we can define the number of defective screws as a discrete random variable. Discrete random variables take a range of integer values. On the other hand, if the hardness of steel is measured from a production process, it will be a continuous random variable taking a range of real values.

RV and probabilities

In the simplest example of a coin toss, there are only two outcomes in the sample space - Head or Tail. A discrete random variable can be defined for this case that will take on the value of 1 for Head and 0 for tail. The corresponding probabilities are:

P (X=0) = 0.5
P (X= 1) = 0.5

RV and probability distribution functions

A discrete probability distribution function (PDF) can be defined for a random variable X that takes a finite number of values $x_1, x_2, \ldots x_n$ with corresponding probabilities $p_1, p_2, \ldots p_n$.

Cumulative distribution function (CDF) is obtained as follows:

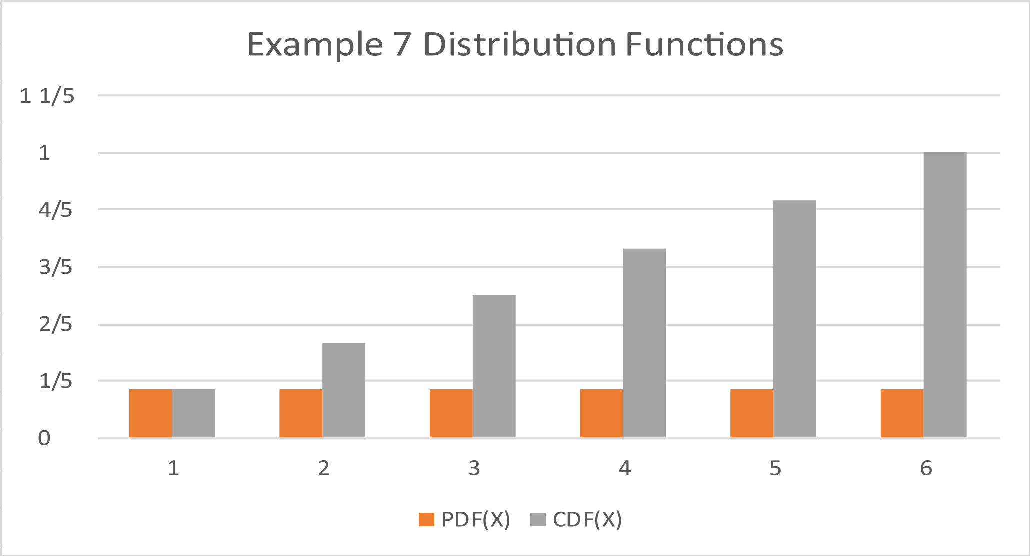
$$P(X \leq x_p) = \sum_{x_j \leq x_p} p_j$$

EXAMPLE-7

Consider a discrete random variable X = Number a fair die throws up. Plot the PDF and CDF for this random variable.

Solution

X	1	2	3	4	5	6
PDF(X)	1/6	1/6	1/6	1/6	1/6	1/6
CDF(X)	1/6	1/3	1/2	2/3	5/6	1



EXAMPLE-8

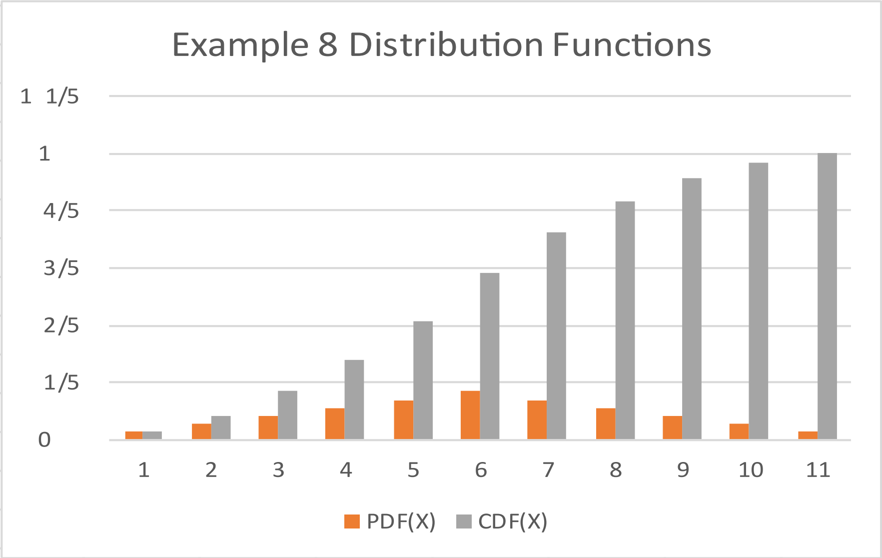
Consider a discrete random variable X = Sum of two numbers thrown up by tossing two fair dies. Plot the PDF and CDF for this random variable.

Solution

The sample space has 6 x 6 = 36 equally likely outcomes. The values of various probabilities are:

Sum = 2 {(1, 1)} = 1 outcome; probability = 1/36
Sum = 3 {(1,2),(2,1)} = 2 outcomes; probability = 2/36
Sum = 4 {(1,3),(2,2),(3,1)} = 3 outcomes; probability = 3/36
Sum = 5 {(1,4),(2,3),(4,1),(3,2)} = 4 outcomes; probability = 4/36
Sum = 6{(1,5),(2,4),(3,3),(4,2),(5,1)} = 5 outcomes; probability = 5/36
Sum = 7{(1,6),(2,5),(3,4),(6,1),(5,2),(4,3) = 6 outcomes; probability = 6/36 and so on

X	2	3	4	5	6	7	8	9	10	11	12
PDF(X)	1/36	1/18	1/12	1/9	5/36	1/6	5/36	1/9	1/12	1/18	1/36
CDF(X)	1/36	1/12	1/6	5/18	5/12	7/12	13/18	5/6	11/12	35/36	1



Geometric probability distribution function

Consider a geometric PDF $(1-p)^{x-1}p$. For probabilities between 0 and 1 and integer x, the distribution function is always positive

This is a geometric series of the form $a, ar, ar^2, \ldots$, with $a = p\ and\ r = 1 - p$
The sum is $\frac{a}{1-r}$; this verifies that the geometric series will add to one

EXAMPLE-9

Consider the problem of tossing a fair coin where the Random Variable X = number of trials, until the first head occurs. Show that the PDF is a geometric progression.

Solution

The probabilities are calculated using independence of events as follows (H= Head, T=Tail)

P (X = 1) = P (H) = 1/2
P (X = 2) = P (TH) = 1/2 x 1/2 = 1/4
P (X = 3) = P (TTH) = 1/2 x 1/2 x 1/2 = 1/8 etc.,

P (X = n) = $(1/2)^n$, n = 1, 2, ….

This is a geometric series with a = p = 1/2, r = 1/2; Sum = (1/2) / {1-(1/2)} = 1.

Binomial probability distribution function

The binomial PDF occurs in games of chance such as rolling a die, quality inspections involving counting the number of defectives, and is defined as $(1-p)^{x-1}p$. The probability, *p*, lies between 0 and 1 and *x* is an integer; the distribution function is always positive

Consider event A with a probability P(A) = p. The Random Variable *X* for the binomial distribution is the number of times event A occurs in *n* trails. If *X* = *x*, the event occurs *x* times and does not occur n-x times.

Because of independence of events, the probability that the event occurs *x* times in *n* trials therefore, is $p^x (1-p)^{n-x}$.
However, the event can occur in several combinations during the *n* trails.

The number of combinations is given by $n!/x!\,(n-x)! = \binom{n}{x}$

Therefore, it follows that the binomial PDF = $\binom{n}{x}$ $p^x (1-p)^{n-x}$

EXAMPLE-10

Consider the probability of obtaining two "four" in rolling a fair die 5 times.

Solution

Using the binomial PDF, p = P (A) = P ("Four") = 1/6; q = 5/6; n = 5, x = 2

The required probability = $\binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$ = (10) x 1/36 x 125/216 = 16 %

Continuous RV

Discrete random variables appear in experiments involving *counting* such as number of defects in a production process. Continuous random variables appear in *measurements* such as length of screws, hardness of steel etc. The equivalent of the PDF becomes a continuous function in the case of a continuous RV.

In other words, we define probability density in a local region

- $f(X = x) = f(x) = P(x \leq X \leq x + dx)/dx$ in the limit as the interval tends to zero
- *f(x)* is the probability density function

- Probability must be positive
  - PDF $f(x) \geq 0 \; \forall \; x$
- $P(-\infty \leq X \leq +\infty) = 1$
  - $\int_{-\infty}^{+\infty} f(x)dx = 1$
- To find the probability that a continuous RV takes a value in the range [*a*,*b*]
  - $P(a \leq X \leq b) = \int_a^b f(x)dx$
- One can also define a cumulative density function $[-\infty, x]$
  - $CDF(x) = \int_{-\infty}^{x} f(\alpha)d\alpha$

EXAMPLE-11

Let X have the probability density function f(x) = 0.75 $(1-x^2)$ if $-1 \leq x \leq 1$ and zero otherwise. Find $P(-{}^1/_2 \leq x \leq {}^1/_2)$

Solution

Required Probability = 0.75 $\int_{-1/2}^{1/2}(1 - v^2)\, dv$ = 68.75%